TO RADIATE OR NOT TO RADIATE

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SUMMARY

Insight into radiation damping of an unbounded medium is developed by addressing the relative contributions of the elastic restoring force and the inertial force at infinity. When the inertial force dominates, radiation damping occurs. When the elastic restoring force dominates, no radiation damping arises. An unbounded medium with a cutoff frequency can also be identified.

KEY WORDS: dynamic stiffness; far field; radiation condition; radiation damping; soil–structure interaction; unbounded medium

1. INTRODUCTION

In a dynamic unbounded medium–structure interaction analysis two different substructures are present: the bounded structure and the unbounded (infinite or semi-infinite) medium. The most striking feature in an unbounded medium, which is never encountered in a bounded medium, is, in general, the presence of *radiation damping* in a frequency domain analysis of a linear elastic system. Mathematically, the corresponding dynamic stiffness at the structure–medium interface describing the relationship between the amplitudes of the displacements and those of the interaction forces is complex for all frequencies. As an example, the horizontal dynamic-stiffness coefficient of a rigid foundation on the surface of a homogeneous half-space is mentioned. As a well-known exception, the horizontal dynamic-stiffness coefficient of a rigid foundation on a horizontal layer fixed at its base exhibits a *cutoff frequency*, below which *no radiation damping* exists, yielding a *real dynamic-stiffness coefficient*. The cutoff frequency is equal to the fundamental frequency of the layer fixed at its base. Thus, below the cutoff frequency the dynamic behaviour is similar to that of a bounded medium with the displacement decaying exponentially, and above the cutoff frequency propagating waves towards infinity exist.

To demonstrate the existence of a cutoff frequency, two simple one-dimensional semi-infinite systems can be addressed: the semi-infinite rod on an elastic foundation (Reference 1, Section 3.2) and the semi-infinite rod with an exponentially increasing area (Reference 2, Section 5.1).

For the semi-infinite rod on an elastic foundation with area A, modulus of elasticity E, mass density ρ and static spring stiffness per unit length k_g (Figure 1), the dynamic-stiffness coefficient $S^\infty(a_0)$ relating the displacement amplitude u_0 to the interaction force amplitude R equals

$$S^\infty(a_0) = K^\infty(k(a_0) + \mathrm{i}a_0\, c(a_0)) \tag{1}$$

with the static-stiffness coefficient

$$K^\infty = \sqrt{EAk_g} \tag{2}$$

and the dimensionless spring and damping coefficients (Figure 2)

$$\text{for } a_0 \leqslant 1, \quad k(a_0) = \sqrt{1 - a_0^2} \tag{3a}$$

$$c(a_0) = 0 \tag{3b}$$

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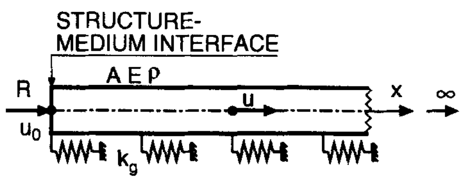

Figure 1. Semi-infinite rod on elastic foundation

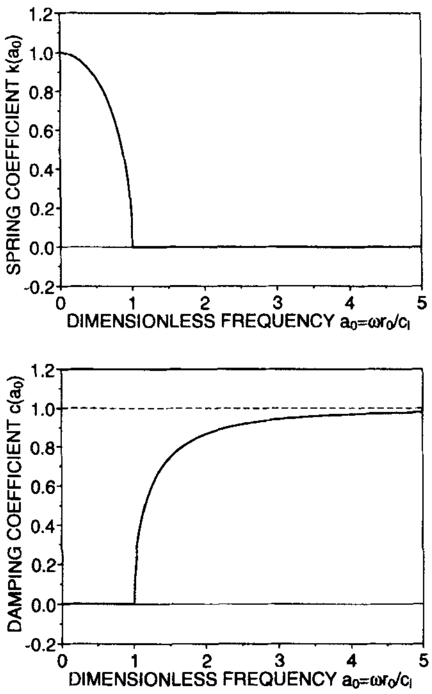

Figure 2. Dynamic-stiffness coefficient of semi-infinite rod on elastic foundation

$$\text{for } a_0 > 1, \quad k(a_0) = 0 \tag{3c}$$

$$c(a_0) = \sqrt{1 - \frac{1}{a_0^2}} \tag{3d}$$

where the dimensionless frequency equals ($r_0 = \sqrt{EA/k_g}$, $c_l = \sqrt{E/\rho}$)

$$a_0 = \frac{\omega r_0}{c_l} \tag{4}$$

Note that the cutoff frequency $a_0 = 1$ corresponds to the fundamental frequency of the system (rigid bar on elastic springs).

For the semi-infinite rod with an exponentially increasing area (Figure 3) $A(x) = A_0 e^{x/f}$ (area A_0 at $x = 0$, length f representing the coordinate x at which the area equals $A_0 e$), modulus of elasticity E and mass density ρ the dynamic-stiffness coefficient equals

$$S^{\infty}(a_0) = K^{\infty}(k(a_0) + \mathrm{i}a_0 c(a_0)) \tag{5}$$

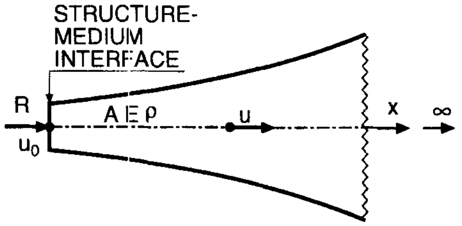

Figure 3. Semi-infinite rod with exponentially increasing area

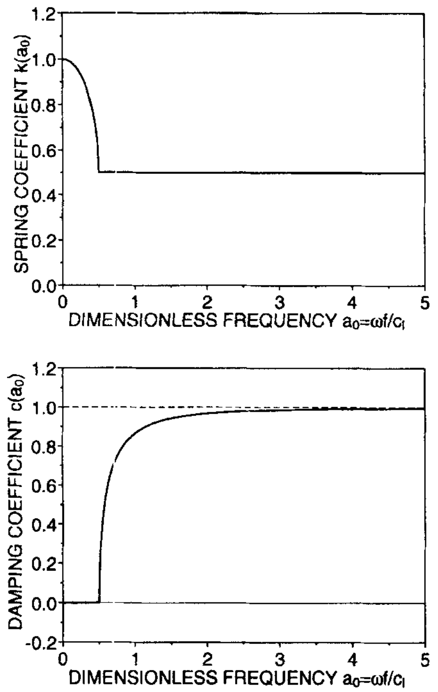

Figure 4. Dynamic-stiffness coefficient of semi-infinite rod with exponentially increasing area

with

$$K^{\infty} = \frac{EA_0}{f} \tag{6}$$

and (Figure 4)

$$\text{for } a_0 \leqslant 0{\cdot}5, \quad k(a_0) = \tfrac{1}{2}\left(1 + \sqrt{1 - 4a_0^2}\right) \tag{7a}$$

$$c(a_0) = 0 \tag{7b}$$

$$\text{for } a_0 > 0{\cdot}5, \quad k(a_0) = \tfrac{1}{2} \tag{7c}$$

$$c(a_0) = \sqrt{1 - \frac{1}{4a_0^2}}. \tag{7d}$$

where ($c_l = \sqrt{E/\rho}$)

$$a_0 = \frac{\omega f}{c_l} \tag{8}$$

Note that for the semi-infinite rod with an exponentially increasing area the cutoff frequency $a_0 = 0{\cdot}5$ does not correspond to a fundamental frequency ($k(a_0 = 0{\cdot}5) \neq 0$).

The following question arise

1. Why does radiation damping for an unbounded medium exist?
2. In which case of an unbounded medium does a cutoff frequency exist?
3. Does an unbounded medium exist for which no radiation damping occurs?

Physical insight can be gained by examining the elastic restoring force and the inertial force of the medium as the radial co-ordinate measured from the structure–medium interface increases towards infinity. The elastic restoring force tends to maintain the original position of the medium, and the inertial force favours movement away from the original position which transmits energy to the adjacent medium. As a wave propagates, for each increase in time an additional part of the unbounded medium (adjacent to the wave front) which was at rest is excited and this requires energy. When the inertial force 'dominates' over the elastic restoring force, this process will continue. As far as the structure's response is concerned, this corresponds to radiation damping. This is the case commonly encountered. However, when the restoring force 'dominates', no radiation damping will arise. Thus, (elastic) *unbounded media exist where no radiation damping appears.* The inertial force is proportional to the square of the excitation frequency. When the elastic restoring force and the inertial force are the 'same' function of the radial co-ordinate, for sufficiently small and large frequencies radiation damping will vanish and be present, respectively. The frequency at which radiation damping first occurs is called the *cutoff frequency*.

The goal of this paper is to establish for a broad class of unbounded media a simple criterion for the existence of radiation damping. It will be assumed that the material properties, i.e., the shear modulus and the mass density, vary as power functions of the radial co-ordinate. For this class of problems the unbounded medium can be assembled from similar finite-element cells. This permits the derivation of the criterion based on the relative contributions of the elastic restoring force and the inertial force of the individual cell, which can be expressed by a suitably defined dimensionless frequency with the radial co-ordinate as a variable. Addressing the asymptotic behaviour of this dimensionless frequency for the radial co-ordinate approaching infinity yields the radiation criterion.

Actually, it is not necessary that the material properties vary as power functions of the radial co-ordinate throughout the unbounded medium from the structure–medium interface to infinity. As only the behaviour at infinity is addressed, it is sufficient that the asymptotic behaviour as the radial co-ordinate approaches infinity for the material properties is governed by power functions. Thus, the behaviour of the materials at finite radial co-ordinates is not restricted.

In Section 2 the dynamic-stiffness matrix of the unbounded medium is constructed by assembling the dynamic-stiffness matrices of the similar finite-element cells. The definition of an appropriate dimensionless frequency results, which leads to the radiation criterion. In Section 3 the out-of-plane motion of a circular cavity embedded in a full-plane is analysed as an example. In Section 4 conclusions are stated.

2. RADIATION CRITERION

To determine the dynamic-stiffness matrix referred to the degrees of freedom on the structure–medium interface of an unbounded medium (Figure 5), concepts of infinite substructuring using similar cells of finite elements[3,4] are applied. The unbounded medium is conceptually divided into similar finite-element cells up to infinity with the similarity centre at O and the constant aspect ratio

$$\gamma = \frac{r_j}{r_{j-1}} \tag{9}$$

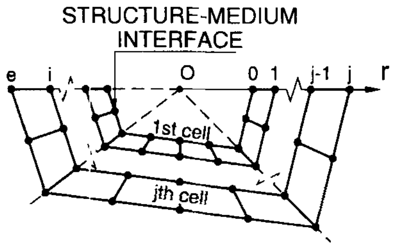

Figure 5. Finite-element cells of unbounded medium

where r is the radial co-ordinate measured from the similarity centre towards infinity. The shear modulus and mass density vary as

$$G(r) = G_0 \left(\frac{r}{r_0}\right)^g \tag{10}$$

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^m \tag{11}$$

G_0 and ρ_0 are the shear modulus and mass density at the structure–medium interface with the radial co-ordinate r_0. The powers g and m are real numbers which can be selected as positive or negative.

The static-stiffness and mass matrices of the jth similar cell are determined by scaling those of the first cell $G_0 r_0^{s-2}[\bar{K}^1]$ and $\rho_0 r_0^s[\bar{M}^1]$, where $[\bar{K}^1]$ and $[\bar{M}^1]$ are dimensionless. The dynamic-stiffness matrix of the jth cell is formulated as

$$[S^j(\omega)] = G_0 r_0^{s-2} \gamma^{(j-1)(g+s-2)} [\bar{K}^1] - \omega^2 \rho_0 r_0^s \gamma^{(j-1)(m+s)} [\bar{M}^1] \tag{12}$$

with the spatial dimension s ($= 2$ or $= 3$). The first and second terms on the right-hand side correspond to the elastic restoring force and inertial force, respectively. Assembling $[S^j(\omega)]$ of all cells ($j = 1, \ldots, \infty$) yields a formal representation of the dynamic property of the unbounded medium with the structure–medium interface characterized by r_0

$$[S(r_0, \omega)] = [K(r_0)] - \omega^2 [M(r_0)] \tag{13}$$

where the matrices of infinite order equal

$$[K(r_0)] = G_0 r_0^{s-2} \begin{bmatrix} [\bar{K}^1_{ii}] & [\bar{K}^1_{ie}] & & \\ [\bar{K}^1_{ei}] & [\bar{K}^1_{ee}] + \gamma^{g+s-2}[\bar{K}^1_{ii}] & \gamma^{g+s-2}[\bar{K}^1_{ie}] & \\ & \gamma^{g+s-2}[\bar{K}^1_{ei}] & \gamma^{g+s-2}[\bar{K}^1_{ee}] + \gamma^{2(g+s-2)}[\bar{K}^1_{ii}] & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

$$= G_0 r_0^{s-2} [\bar{K}] \tag{14}$$

and

$$[M(r_0)] = \rho_0 r_0^s \begin{bmatrix} [\bar{M}^1_{ii}] & [\bar{M}^1_{ie}] & & \\ [\bar{M}^1_{ei}] & [\bar{M}^1_{ee}] + \gamma^{m+s}[\bar{M}^1_{ii}] & \gamma^{m+s}[\bar{M}^1_{ie}] & \\ & \gamma^{m+s}[\bar{M}^1_{ei}] & \gamma^{m+s}[\bar{M}^1_{ee}] + \gamma^{2(m+s)}[\bar{M}^1_{ii}] & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

$$= \rho_0 r_0^s [\bar{M}] \tag{15}$$

The subscripts i and e refer to the interior and exterior boundaries of the cells.

Analogously, assembling $[S^j(\omega)]$ of all but the first cell ($j = 2, \ldots, \infty$) yields

$$[S(r_1, \omega)] = [K(r_1)] - \omega^2 [M(r_1)] \tag{16}$$

where

$$[K(r_1)] = G_0 \frac{r_1^{g+s-2}}{r_0^g} [\bar{K}] \tag{17}$$

$$[M(r_1)] = \rho_0 \frac{r_1^{m+s}}{r_0^m} [\bar{M}] \tag{18}$$

Substituting equations (17) and (18) in equation (16) leads to

$$[S(r_1, \omega)] = G_0 \frac{r_1^{g+s-2}}{r_0^g} ([\bar{K}] - a^2 [\bar{M}]) \tag{19}$$

where the dimensionless frequency for r_1 is defined as

$$a = \frac{\omega}{c_{s0}} \frac{r_1^{1-(g/2)+(m/2)}}{r_0^{-(g/2)+(m/2)}} \tag{20}$$

This procedure can be generalized to any structure–medium interface with the radial co-ordinate r by assembling $[S^j(\omega)]$ of all cells located between this structure–medium interface and infinity resulting in

$$[S(r, \omega)] = G_0 \frac{r^{g+s-2}}{r_0^g} ([\bar{K}] - a^2 [\bar{M}]) \tag{21}$$

with

$$a = \frac{\omega}{c_{s0}} \frac{r^{1-(g/2)+(m/2)}}{r_0^{-(g/2)+(m/2)}} \tag{22}$$

Note that the dimensionless frequency a is a function of the radial co-ordinate r. Conceptually, eliminating all degrees of freedom with the exception of those on the structure–medium interface with r in equation (21) yields the corresponding dynamic-stiffness matrix of the unbounded medium $[S^\infty(r, \omega)]$. The latter is proportional to $G_0 r^{g+s-2}/r_0^g$ and is a function of a ($= [S^\infty(a)]$).

The radiation criterion is discussed next. As the relative contributions of the elastic restoring force and the inertial force are the same for the unbounded medium expressed as matrices of infinite order (equations (14) and (15)) and for the individual cell (equation (12)), it is sufficient to address one cell to examine if radiation damping occurs. The dynamic-stiffness matrix of a cell with the radial co-ordinate r at the interior boundary specified in equation (12) is written as (with superscript c for cell)

$$[S^c(\omega)] = [K^c] - \omega^2 [M^c] \tag{23}$$

with the static-stiffness and mass matrices of the cell

$$[K^c] = G_0 r_0^{s-2} \left(\frac{r}{r_0}\right)^{g+s-2} [\bar{K}^1] \tag{24}$$

$$[M^c] = \rho_0 r_0^s \left(\frac{r}{r_0}\right)^{m+s} [\bar{M}^1] \tag{25}$$

$[K^c]$ multiplied by the displacement amplitudes $\{u(\omega)\}$ represents the amplitudes of the elastic restoring force and $-\omega^2 [M^c] \{u(\omega)\}$ the amplitudes of the inertial force of the cell. $[S^c(\omega)]$ can be reformulated as

$$[S^c(\omega)] = G_0 r_0^{s-2} \left(\frac{r}{r_0}\right)^{g+s-2} ([\bar{K}^1] - a^2 [\bar{M}^1]) \tag{26}$$

with the dimensionless frequency a specified in equation (22). a is a dimensionless variable describing the relative contributions of the inertial part and the elastic restoring part to the dynamic-stiffness matrix as a function of r and ω.

The boundary condition at infinity must be addressed in an unbounded medium. For $r \to \infty$ and $\omega \neq 0$, three cases exist, which are determined by the power of r in a (equation (22)).

For

$$1 - \frac{g}{2} + \frac{m}{2} > 0 \tag{27}$$

$a \to \infty$ for $r \to \infty$ applies. The inertial force will always dominate over the elastic restoring force for any ω. Waves propagating towards infinity exist, and thus radiation damping will always be present.

For

$$1 - \frac{g}{2} + \frac{m}{2} < 0 \tag{28}$$

$a \to 0$ for $r \to \infty$ holds. The elastic restoring force will always dominate for any ω. In this case, only evanescent waves which do not propagate towards infinity arise, and radiation damping will never exist. This can be achieved in an unbounded medium whose shear modulus increases sufficiently in the radial direction and/or whose mass density decreases sufficiently, so that equation (28) is satisfied.

For the intermediate case

$$1 - \frac{g}{2} + \frac{m}{2} = 0 \tag{29}$$

a is independent of r, but still depends on ω. For sufficiently small and large frequencies the elastic restoring force and the inertial force dominate, respectively. Thus, a cutoff frequency exists below which no radiation damping occurs.

Equations (27)–(29) describe the *radiation criterion*.

For the homogeneous half-space mentioned in the Introduction ($g = m = 0$), equation (27) applies and radiation damping occurs for all frequencies. For the semi-infinite rod on an elastic foundation, conceptionally the similarity centre can be placed at infinity. The a of the individual cell does not depend on the radial co-ordinate and thus a cutoff frequency will occur.

3. OUT-OF-PLANE MOTION OF CIRCULAR CAVITY EMBEDDED IN FULL-PLANE

A simple one-dimensional wave propagation problem, where an analytical solution exists for the shear modulus varying in the radial direction, is used to investigate the three cases of wave behaviour. The out-of-plane motion of a circular cavity of radius r_0 embedded in a full-plane with a uniform displacement u_0 enforced on its wall, the structure–medium interface, is discussed (Figure 6). Symmetric waves exist. The

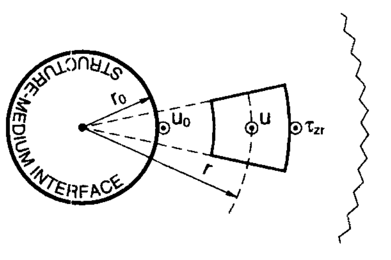

Figure 6. Out-of-plane motion of circular cavity embedded in full-plane with symmetric waves

shear modulus G varies in the radial direction as

$$G(r) = G_0 \left(\frac{r}{r_0}\right)^g \tag{30}$$

and the mass density ρ is constant.

Substituting the stress–displacement relationship

$$\tau_{zr} = G(r) u_{,r} \tag{31}$$

into the equilibrium equation

$$\tau_{zr,r} + \frac{1}{r}\tau_{zr} - \rho\ddot{u} = 0 \tag{32}$$

leads to the equation of motion

$$G(r)u_{,rr} + \left(G(r)_{,r} + \frac{1}{r}G(r)\right)u_{,r} - \rho\ddot{u} = 0 \tag{33}$$

A solution is derived in the frequency domain. Substituting equation (30) in equation (33) formulated in the frequency domain yields

$$r^2 u(\omega)_{,rr} + (g+1)\, r u(\omega)_{,r} + \frac{\omega^2 r^{2-g}}{c_{s0}^2 r_0^{-g}} u(\omega) = 0 \tag{34}$$

with the shear-wave velocity at the structure–medium interface

$$c_{s0} = \sqrt{\frac{G_0}{\rho}} \tag{35}$$

To derive a solution for equation (34), the following two transformations of variables are performed (Reference 5, p. 440)

$$u(\omega) = v(a)\left(\frac{r}{r_0}\right)^{-g/2} \tag{36}$$

$$a = \frac{\omega}{c_{s0}} \frac{r^{1-g/2}}{r_0^{-g/2}} \tag{37}$$

For the independent variable r the dimensionless frequency a is used. Substituting in equation (34) leads to

$$a^2 v(a)_{,aa} + a v(a)_{,a} + ((\lambda a)^2 - (\lambda - 1)^2) v(a) = 0 \tag{38}$$

with

$$\lambda = \frac{2}{2-g} \tag{39}$$

This is the Bessel differential equation. Note that equation (37) is the same as the definition of a in equation (22) with $m = 0$.

The behaviour of equation (38) is governed by the boundary condition specified for a as $r \to \infty$ and thus the power of r in equation (37). The same three cases as in Section 2 appear.

For $1 - 0{\cdot}5g > 0$ (equation (27) with $m = 0$), i.e. $g < 2$, $\lambda a \to \infty$ applies for $r \to \infty$ as verified from equation (37). The solution for the displacement follows from equations (38) and (36) as

$$u(\omega) = \left(\frac{r}{r_0}\right)^{-g/2} (c_1 H^{(1)}_{|\lambda-1|}(\lambda a) + c_2 H^{(2)}_{|\lambda-1|}(\lambda a)) \tag{40}$$

$H^{(1)}_{|\lambda-1|}$ and $H^{(2)}_{|\lambda-1|}$ are the first and second kind Hankel functions of order $|\lambda - 1|$, and c_1, c_2 are the integration constants. The corresponding asymptotic behaviour for $\lambda a \to \infty$ equals

$$H^{(1)}_{|\lambda-1|}(\lambda a) \approx \sqrt{\frac{2}{\pi}\frac{1}{\lambda a}}\, e^{+i(\lambda a - (|\lambda - 1|\pi/2) - (\pi/4))} \tag{41a}$$

$$H^{(2)}_{|\lambda-1|}(\lambda a) \approx \sqrt{\frac{2}{\pi}\frac{1}{\lambda a}}\, e^{-i(\lambda a - (|\lambda - 1|\pi/2) - (\pi/4))} \tag{41b}$$

Equations (41a) and (41b) correspond to incoming and outgoing waves, respectively. As only outgoing waves can exist, $c_1 = 0$ is obtained. This results in

$$u(\omega) = c\left(\frac{r}{r_0}\right)^{-g/2} H^{(2)}_{|\lambda-1|}(\lambda a) \tag{42}$$

where the subscript 2 in the integration constant is omitted.

The dynamic-stiffness coefficient $S^{\infty}(\omega)$ relates the displacement amplitude $u_0(\omega)$ to the interaction force amplitude

$$R(\omega) = -2\pi r_0 G_0\, u(\omega)_{,r}|_{r=r_0} \tag{43}$$

as

$$R(\omega) = S^{\infty}(\omega)\, u_0(\omega) \tag{44}$$

Performing the derivative $u(\omega)_{,r}$ in equation (42) using

$$\lambda a\, H^{(2)}_{|\lambda-1|}(\lambda a)_{,\lambda a} = |\lambda - 1| H^{(2)}_{|\lambda-1|}(\lambda a) - \lambda a H^{(2)}_{|\lambda-1|+1}(\lambda a) \tag{45}$$

yields

$$S^{\infty}(a_0) = 2\pi G_0 \left(\frac{\lambda - 1 - |\lambda - 1|}{\lambda} + \frac{|\lambda|}{\lambda} a_0 \frac{H^{(2)}_{|\lambda-1|+1}(\lambda a_0)}{H^{(2)}_{|\lambda-1|}(\lambda a_0)}\right) \tag{46}$$

with the dimensionless frequency at the structure–medium interface ($r = r_0$)

$$a_0 = \frac{\omega r_0}{c_{s0}} \tag{47}$$

The static-stiffness coefficient follows for the limit $\omega \to 0$ as

$$K^{\infty} = \begin{cases} 2\pi G_0 g, & g > 0 \\ 0, & g \leqslant 0 \end{cases} \tag{48}$$

The dynamic-stiffness coefficient is non-dimensionalized with $2\pi G_0$ and decomposed as

$$S^{\infty}(a_0) = 2\pi G_0(k(a_0) + i a_0 c(a_0)) \tag{49}$$

The dimensionless spring and damping coefficients are plotted for $g = 0$ (homogeneous case) and $g = 1$ (linear increase of G) in Figure 7. As expected, radiation damping reflected by $c(a_0) > 0$ occurs for all non-zero frequencies. The damping ratio

$$\zeta(a_0) = \frac{a_0\, c(a_0)}{2k(a_0)} \tag{50}$$

is shown in Figure 8. Although $c(a_0 \to \infty)$ for the two cases are the same, large differences in the damping ratio exist for all a_0.

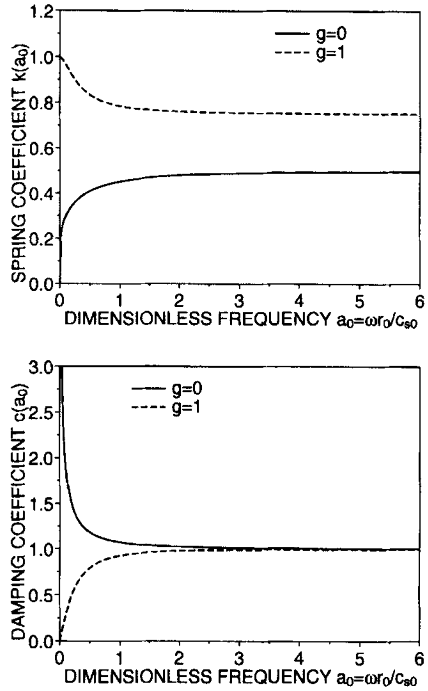

Figure 7. Dynamic-stiffness coefficient of out-of-plane motion of circular cavity for constant and linear increase of shear modulus in radial direction

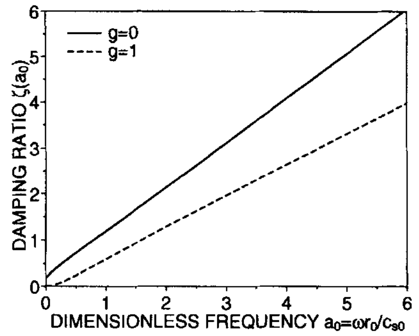

Figure 8. Damping ratio of out-of-plane motion of circular cavity for constant and linear increase of shear modulus in radial direction

For $1 - 0{\cdot}5g < 0$ (equation (28) with $m = 0$), i.e. $g > 2$, $-\lambda a \to 0$ applies for $r \to \infty$ as verified from equation (37). The solution for the displacement follows from equations (38) and (36) as

$$u(\omega) = \left(\frac{r}{r_0}\right)^{-g/2} (c_1 Y_{1-\lambda}(-\lambda a) + c_2 J_{1-\lambda}(-\lambda a)) \tag{51}$$

$Y_{1-\lambda}$ and $J_{1-\lambda}$ are the second and first kind Bessel functions of order $1 - \lambda$, and c_1, c_2 are the integration constants. As $|Y_{1-\lambda}(-\lambda a \to 0)| \to \infty$, $c_1 = 0$ follows. This results in

$$u(\omega) = c\left(\frac{r}{r_0}\right)^{-g/2} J_{1-\lambda}(-\lambda a) \tag{52}$$

The dynamic-stiffness coefficient defined in equations (43) and (44) equals

$$S^{\infty}(\omega) = 2\pi G_0 \left(g - a_0 \frac{J_{2-\lambda}(-\lambda a_0)}{J_{1-\lambda}(-\lambda a_0)} \right) \tag{53}$$

$S^{\infty}(a_0)$ is real for all frequencies. The system exhibits resonances. This unbounded medium thus behaves similarly as a bounded medium. No radiation damping occurs. The dynamic-stiffness coefficient non-dimensionalized with $2\pi G_0$ is plotted for $g = 3$ in Figure 9.

For $1 - 0{\cdot}5g = 0$ (equation (29) with $m = 0$), i.e. $g = 2$, $a = a_0 = \omega r_0/c_{s0}$ is independent of r. The transformation of the independent variable r to a defined in equation (37) breaks down. Equation (34) is written as

$$r^2 u(\omega)_{,rr} + 3ru(\omega)_{,r} + a_0^2 u(\omega) = 0 \tag{54}$$

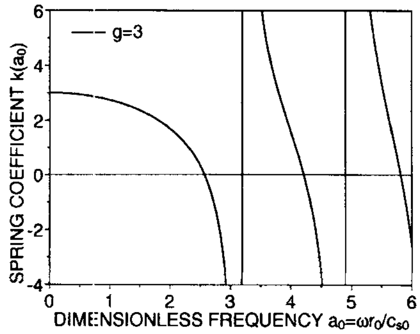

Figure 9. Dynamic-stiffness coefficient of out-of-plane motion of circular cavity for strong increase of shear modulus in radial direction

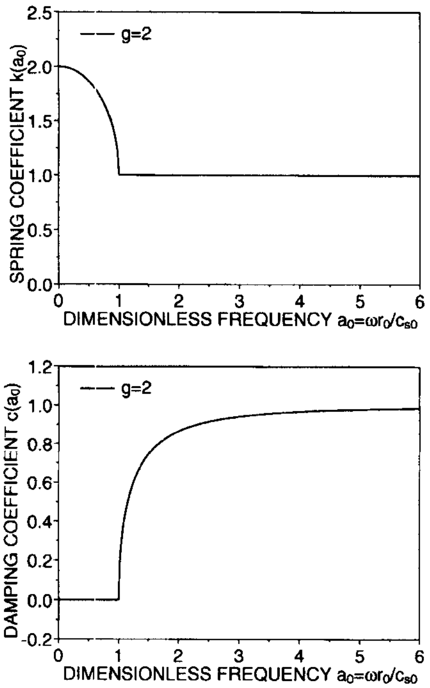

Figure 10. Dynamic-stiffness coefficient of out-of-plane motion of circular cavity for parabolic increase of shear modulus in radial direction

The solution of equation (54) equals

$$u(\omega) = c_1 r^{-1+\sqrt{1-a_0^2}} + c_2 r^{-1-\sqrt{1-a_0^2}} \tag{55}$$

Addressing the static case ($\omega = 0$), the first term corresponds to a constant displacement. Thus, $c_1 = 0$ is enforced. The solution equals

$$u(\omega) = c\, r^{-1-\sqrt{1-a_0^2}} \tag{56}$$

The dynamic-stiffness coefficient $S^{\infty}(a_0)$ follows from equations (43) and (44) as

$$S^{\infty}(a_0) = 2\pi G_0 \left(1 + \sqrt{1 - a_0^2}\right) \tag{57}$$

Below the cutoff frequency $\omega = c_{s0}/r_0$ no radiation damping occurs. $S^{\infty}(a_0)$ is non-dimensionalized with $2\pi G_0$. The dimensionless spring and damping coefficients are plotted in Figure 10. An analogous behaviour as for the semi-infinite rod with an exponentially increasing area occurs.

4. CONCLUSIONS

The behaviour concerning radiation damping of an unbounded medium is determined by the relative contributions of the elastic restoring force and the inertial force at infinity. Three cases are identified:

1. When the inertial force dominates over the elastic restoring force, radiation damping occurs for all frequencies.
2. When the elastic restoring force dominates over the inertial force, no radiation damping occurs for any frequency.
3. When the relative contributions of the elastic restoring force and the inertial force only depend on the frequency and not on the radial co-ordinate, a cutoff frequency exists.

For material properties varying as power functions of the radial co-ordinate, the three cases follow from the difference of the powers of the shear modulus and the mass density. In particular, by selecting a sufficiently large increase in the radial direction of the shear modulus or a sufficiently large decrease of the mass density, the unbounded medium will behave similarly as a bounded medium exhibiting resonances and no radiation damping.

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